

# DIFFERENCE BASES IN FINITE ABELIAN GROUPS

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**ABSTRACT.** A subset  $B$  of a group  $G$  is called a *difference basis* of  $G$  if each element  $g \in G$  can be written as the difference  $g = ab^{-1}$  of some elements  $a, b \in B$ . The smallest cardinality  $|B|$  of a difference basis  $B \subset G$  is called the *difference size* of  $G$  and is denoted by  $\Delta[G]$ . The fraction  $\mathfrak{d}[G] := \frac{\Delta[G]}{\sqrt{|G|}}$  is called the *difference characteristic* of  $G$ . Using properties of the Galois rings, we prove recursive upper bounds for the difference sizes and characteristics of finite Abelian groups. In particular, we prove that for a prime number  $p \geq 11$ , any finite Abelian  $p$ -group  $G$  has difference characteristic  $\mathfrak{d}[G] < \frac{\sqrt{p}-1}{\sqrt{p}-3} \cdot \sup_{k \in \mathbb{N}} \mathfrak{d}[C_{p^k}] < \sqrt{2} \cdot \frac{\sqrt{p}-1}{\sqrt{p}-3}$ . Also we calculate the difference sizes of all Abelian groups of cardinality  $< 96$ .

## 1. INTRODUCTION

A subset  $B$  of a group  $G$  is called a *difference basis* for a subset  $A \subset G$  if each element  $a \in A$  can be written as  $a = xy^{-1}$  for some  $x, y \in B$ . The smallest cardinality of a difference basis for  $A$  is called the *difference size* of  $A$  and is denoted by  $\Delta[A]$ . For example, the set  $\{0, 1, 4, 6\}$  is a difference basis for the interval  $A = [-6, 6] \cap \mathbb{Z}$  witnessing that  $\Delta[A] \leq 4$ . The difference size is subadditive in the sense that  $\Delta[A \cup B] < \Delta[A] + \Delta[B]$  for any non-empty finite subsets  $A, B$  of a group  $G$  (see Proposition 4.1).

The definition of a difference basis  $B$  for a set  $A$  in a group  $G$  implies that  $|A| \leq |B|^2$  and hence  $\Delta[A] \geq \sqrt{|A|}$ . The fraction

$$\mathfrak{d}[A] := \frac{\Delta[A]}{\sqrt{|A|}} \geq 1$$

is called the *difference characteristic* of  $A$ . The difference characteristic is submultiplicative in the sense that  $\mathfrak{d}[G] \leq \mathfrak{d}[H] \cdot \mathfrak{d}[G/H]$  for any normal subgroup  $H$  of a finite group  $G$ , see [1, 1.1].

In this paper we are interested in evaluating the difference characteristics of finite Abelian groups. In fact, this problem has been studied in the literature. A fundamental result in this area is due to Kozma and Lev [8], who proved (using the classification of finite simple groups) that each finite group  $G$  has difference characteristic  $\mathfrak{d}[G] \leq \frac{4}{\sqrt{3}} \approx 2.3094$ . For finite cyclic groups  $G$  the upper bound  $\frac{4}{\sqrt{3}}$  can be lowered to  $\mathfrak{d}[G] \leq \frac{3}{2}$  (and even to  $\mathfrak{d}[G] < \frac{2}{\sqrt{3}}$  if  $|G| \geq 2 \cdot 10^{15}$ ), see [2]. In this paper we continue investigations started in [2] and shall give some lower and upper bounds for the difference characteristics of finite Abelian groups. In some cases (for example, for Abelian  $p$ -groups with  $p \geq 11$ ) our upper bounds are better than the general upper bound  $\frac{4}{\sqrt{3}}$  of Kozma and Lev.

In particular, in Theorem 6.3 we prove that for any prime number  $p \geq 11$ , any finite Abelian  $p$ -group  $G$  has difference characteristic

$$\mathfrak{d}[G] \leq \frac{\sqrt{p}-1}{\sqrt{p}-3} \cdot \sup_{k \in \mathbb{N}} \mathfrak{d}[C_{p^k}] < \frac{\sqrt{p}-1}{\sqrt{p}-3} \cdot \sqrt{2}.$$

These results are obtained exploiting a structure of a Galois ring on the groups  $C_{p^k}^r$ . Here  $C_n = \{z \in \mathbb{C} : z^n = 1\}$  is the cyclic group of order  $n$ . The group  $C_n$  is isomorphic to the additive group of the ring  $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ .

## 2. KNOWN RESULTS

In this section we recall some known results on difference bases in finite groups. The following fundamental fact was proved by Kozma and Lev [8].

**Theorem 2.1** (Kozma, Lev). *Each finite group  $G$  has difference characteristic  $\mathfrak{d}[G] \leq \frac{4}{\sqrt{3}}$ .*

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For a real number  $x$  we put

$$\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\} \text{ and } \lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}.$$

The following proposition is proved in [1, 1.1].

**Proposition 2.2.** *Let  $G$  be a finite group. Then*

- (1)  $\frac{1+\sqrt{4|G|-3}}{2} \leq \Delta[G] \leq \lceil \frac{|G|+1}{2} \rceil$ ,
- (2)  $\Delta[G] \leq \Delta[H] \cdot \Delta[G/H]$  and  $\delta[G] \leq \delta[H] \cdot \delta[G/H]$  for any normal subgroup  $H \subset G$ ;
- (3)  $\Delta[G] \leq |H| + |G/H| - 1$  for any subgroup  $H \subset G$ ;

Finite groups  $G$  with  $\Delta[G] = \lceil \frac{|G|+1}{2} \rceil$  were characterized in [1] as follows.

**Theorem 2.3** (Banakh, Gavrylkiv, Nykyforchyn). *For a finite group  $G$*

- (i)  $\Delta[G] = \lceil \frac{|G|+1}{2} \rceil > \frac{|G|}{2}$  if and only if  $G$  is isomorphic to one of the groups:  
 $C_1, C_2, C_3, C_4, C_2 \times C_2, C_5, D_6, (C_2)^3$ ;
- (ii)  $\Delta[G] = \frac{|G|}{2}$  if and only if  $G$  is isomorphic to one of the groups:  $C_6, C_8, C_4 \times C_2, D_8, Q_8$ .

In this theorem by  $D_{2n}$  we denote the dihedral group of cardinality  $2n$  and by  $Q_8$  the 8-element group of quaternion units. In [1] the difference size  $\Delta[G]$  was calculated for all groups  $G$  of cardinality  $|G| \leq 13$ .

TABLE 1. Difference sizes of groups of order  $\leq 13$

$G$ :	$C_2$	$C_3$	$C_5$	$C_4$	$C_2 \times C_2$	$C_6$	$D_6$	$C_8$	$C_2 \times C_4$	$D_8$	$Q_8$	$(C_2)^3$
$\Delta[G]$	2	2	3	3	3	3	4	4	4	4	4	5
$G$ :	$C_7$	$C_{11}$	$C_{13}$	$C_9$	$C_3 \times C_3$	$C_{10}$	$D_{10}$	$C_{12}$	$C_2 \times C_6$	$D_{12}$	$A_4$	$C_3 \rtimes C_4$
$\Delta[G]$	3	4	4	4	4	4	4	4	5	5	5	5

An important role in evaluating the difference sizes of cyclic groups is due to difference sizes of the order-intervals  $[1, n] \cap \mathbb{Z}$  in the additive group  $\mathbb{Z}$  of integer numbers. For a natural number  $n \in \mathbb{N}$  by  $\Delta[n]$  we denote the difference size of the order interval  $[1, n] \cap \mathbb{Z}$  and by  $\delta[n] := \frac{\Delta[n]}{\sqrt{n}}$  its difference characteristic. The asymptotics of the sequence  $(\delta[n])_{n=1}^\infty$  was studied by Rédei and Rényi [10], Leech [7] and Golay [6] who eventually proved that

$$\sqrt{2 + \frac{4}{3\pi}} < \sqrt{2 + \max_{0 < \varphi < 2\pi} \frac{2 \sin(\varphi)}{\varphi + \pi}} \leq \lim_{n \rightarrow \infty} \delta[n] = \inf_{n \in \mathbb{N}} \delta[n] \leq \delta[6166] = \frac{128}{\sqrt{6166}} < \delta[6] = \sqrt{\frac{8}{3}}.$$

In [2] the difference sizes of the order-intervals  $[1, n] \cap \mathbb{Z}$  were applied to give upper bounds for the difference sizes of finite cyclic groups.

**Proposition 2.4.** *For every  $n \in \mathbb{N}$  we get the upper bound  $\Delta[C_n] \leq \Delta[\lceil \frac{n-1}{2} \rceil]$ , which implies that*

$$\limsup_{n \rightarrow \infty} \delta[C_n] \leq \frac{1}{\sqrt{2}} \inf_{n \in \mathbb{N}} \delta[n] \leq \frac{64}{\sqrt{3083}} < \frac{2}{\sqrt{3}}.$$

The following facts on the difference sizes of cyclic groups were proved in [2].

**Theorem 2.5** (Banakh, Gavrylkiv). *For any  $n \in \mathbb{N}$  the cyclic group  $C_n$  has the difference characteristic:*

- (1)  $\delta[C_n] \leq \delta[C_4] = \frac{3}{2}$ ;
- (2)  $\delta[C_n] \leq \delta[C_2] = \delta[C_8] = \sqrt{2}$  if  $n \neq 4$ ;
- (3)  $\delta[C_n] \leq \frac{12}{\sqrt{73}} < \sqrt{2}$  if  $n \geq 9$ ;
- (4)  $\delta[C_n] \leq \frac{24}{\sqrt{293}} < \frac{12}{\sqrt{73}}$  if  $n \geq 9$  and  $n \neq 292$ ;
- (5)  $\delta[C_n] < \frac{2}{\sqrt{3}}$  if  $n \geq 2 \cdot 10^{15}$ .

For some special numbers  $n$  we have more precise upper bounds for  $\Delta[C_n]$ . We recall that a number  $q$  is a *prime power* if  $q = p^k$  for some prime number  $p$  and some  $k \in \mathbb{N}$ .

The following theorem was derived in [2] from the classical results of Singer [13], Bose, Chowla [4], [5] and Rusza [12].

**Theorem 2.6.** *Let  $p$  be a prime number and  $q$  be a prime power. Then*

- (1)  $\Delta[C_{q^2+q+1}] = q + 1$ ;

- (2)  $\Delta[C_{q^2-1}] \leq q-1 + \Delta[C_{q-1}] \leq q-1 + \frac{3}{2}\sqrt{q-1}$ ;  
(3)  $\Delta[C_{p^2-p}] \leq p-3 + \Delta[C_p] + \Delta[C_{p-1}] \leq p-3 + \frac{3}{2}(\sqrt{p} + \sqrt{p-1})$ .

The following table of difference sizes of cyclic groups  $C_n$  for  $n \leq 100$  is taken from [2].

TABLE 2. Difference sizes and characteristics of cyclic groups  $C_n$  for  $n \leq 100$

$n$	$\Delta[C_n]$	$\bar{\delta}[C_n]$	$n$	$\Delta[C_n]$	$\bar{\delta}[C_n]$	$n$	$\Delta[C_n]$	$\bar{\delta}[C_n]$	$n$	$\Delta[C_n]$	$\bar{\delta}[C_n]$
1	1	1	26	6	1.1766...	51	8	1.1202...	76	10	1.1470...
2	2	1.4142...	27	6	1.1547...	52	9	1.2480...	77	10	1.1396...
3	2	1.1547...	28	6	1.1338...	53	9	1.2362...	78	10	1.1322...
4	3	1.5	29	7	1.2998...	54	9	1.2247...	79	10	1.1250...
5	3	1.3416...	30	7	1.2780...	55	9	1.2135...	80	11	1.2298...
6	3	1.2247...	31	6	1.0776...	56	9	1.2026...	81	11	1.2222...
7	3	1.1338...	32	7	1.2374...	57	8	1.0596...	82	11	1.2147...
8	4	1.4142...	33	7	1.2185...	58	9	1.1817...	83	11	1.2074...
9	4	1.3333...	34	7	1.2004...	59	9	1.1717...	84	11	1.2001...
10	4	1.2649...	35	7	1.1832...	60	9	1.1618...	85	11	1.1931...
11	4	1.2060...	36	7	1.1666...	61	9	1.1523...	86	11	1.1861...
12	4	1.1547...	37	7	1.1507...	62	9	1.1430...	87	11	1.1793...
13	4	1.1094...	38	8	1.2977...	63	9	1.1338...	88	11	1.1726...
14	5	1.3363...	39	7	1.1208...	64	9	1.125	89	11	1.1659...
15	5	1.2909...	40	8	1.2649...	65	9	1.1163...	90	11	1.1595...
16	5	1.25	41	8	1.2493...	66	10	1.2309...	91	10	1.0482...
17	5	1.2126...	42	8	1.2344...	67	10	1.2216...	92	11	1.1468...
18	5	1.1785...	43	8	1.2199...	68	10	1.2126...	93	12	1.2443...
19	5	1.1470...	44	8	1.2060...	69	10	1.2038...	94	12	1.2377...
20	6	1.3416...	45	8	1.1925...	70	10	1.1952...	95	12	1.2311...
21	5	1.0910...	46	8	1.1795...	71	10	1.1867...	96	12	1.2247...
22	6	1.2792...	47	8	1.1669...	72	10	1.1785...	97	12	1.2184...
23	6	1.2510...	48	8	1.1547...	73	9	1.0533...	98	12	1.2121...
24	6	1.2247...	49	8	1.1428...	74	10	1.1624...	99	12	1.2060...
25	6	1.2	50	8	1.1313...	75	10	1.1547...	100	12	1.2

### 3. A LOWER BOUND FOR THE DIFFERENCE SIZE

In this section we prove a simple lower bound for the difference size of an arbitrary finite set in a group. This lower bound improves the lower bound given in Proposition 2.2(1). For a group  $G$  by  $1_G$  we denote the unique idempotent of  $G$ .

**Theorem 3.1.** *Each finite subset  $A$  of a group  $G$  has difference size*

$$\Delta[A] \geq \frac{1 + \sqrt{4|A_{>2}| + 8|A_2| + 1}}{2} \geq \frac{1 + \sqrt{4|A_{>1}| + 1}}{2},$$

where  $A_{>2} = \{a \in A : a^{-1} \neq a\}$ ,  $A_2 = \{a \in A : a^{-1} = a \neq 1_G\}$  and  $A_{>1} = \{a \in A : a \neq 1_G\}$ .

*Proof.* Take a difference basis  $B \subset G$  for the set  $A$  of cardinality  $|B| = \Delta[A]$  and consider the map  $\xi : B \times B \rightarrow G$ ,  $\xi : (x, y) \mapsto xy^{-1}$ . Observe that for the unit  $1_G$  of the group  $G$  the preimage  $\xi^{-1}(1_G)$  coincides with the diagonal  $\{(x, y) \in B \times B : x = y\}$  of the square  $B \times B$  and hence has cardinality  $|\xi^{-1}(e)| = |B|$ . Observe also that for any element  $g \in A_2 = \{a \in A : a^{-1} = a \neq 1_G\}$  and any  $(x, y) \in \xi^{-1}(g)$ , we get  $yx^{-1} = (xy^{-1})^{-1} = g^{-1} = g$ , which implies that  $|\xi^{-1}(g)| \geq 2$ . Then

$$|B|^2 = |B \times B| \geq |\xi^{-1}(1_G)| + \sum_{a \in A_2} |\xi^{-1}(a)| + \sum_{a \in A_{>2}} |\xi^{-1}(a)| \geq |B| + 2|A_2| + |A_{>2}|$$

and hence

$$\Delta[G] = |B| \geq \frac{1 + \sqrt{1 + 4|A_{>2}| + 8|A_2|}}{2} \geq \frac{1 + \sqrt{1 + 4|A_{>1}|}}{2}$$

as  $A_{>2} \cup A_2 = A_{>1}$ . □

**Corollary 3.2.** *Each finite group  $G$  has difference size  $\Delta[G] \geq \frac{1+\sqrt{4|G|+4|G_2|-3}}{2}$ , where  $G_2 = \{g \in G : g^{-1} = g \neq 1_G\}$  is the set of elements of order 2 in  $G$ .*

#### 4. THE SUBADDITIVITY AND SUBMULTIPLICATIVITY OF THE DIFFERENCE SIZE

In this section we prove two properties of the difference size called the subadditivity and the submultiplicativity.

**Proposition 4.1.** *For any non-empty finite subsets  $A, B$  of a group  $G$  we get  $\Delta[A \cup B] \leq \Delta[A] + \Delta[B] - 1$ .*

*Proof.* Given non-empty sets  $A, B \subset G$ , find difference bases  $D_A$  and  $D_B$  for the sets  $A, B$  of cardinality  $|D_A| = \Delta[A]$  and  $|D_B| = \Delta[B]$ . Taking any point  $d \in D_A$  and replacing  $D_A$  by its shift  $D_A d^{-1}$ , we can assume that the unit  $1_G$  of the group  $G$  belongs to  $D_A$ . By the same reason, we can assume that  $1_G \in D_B$ . The union  $D = D_A \cup D_B$  is a difference basis for  $A \cup B$ , witnessing that

$$\Delta[A \cup B] \leq |D| \leq |D_A| + |D_B| - 1 = \Delta[A] + \Delta[B] - 1.$$

□

**Proposition 4.2.** *Let  $h : G \rightarrow H$  be a surjective homomorphism of groups with finite kernel  $K$ . For any non-empty finite subset  $A \subset H$  we get  $\Delta[h^{-1}(A)] \leq \Delta[A] \cdot \Delta[K]$ .*

*Proof.* Given a non-empty finite subset  $A \subset H$ , find a difference basis  $D_A$  for the set  $A$  of cardinality  $|D_A| = \Delta[A]$ . Also fix a difference basis  $D_K$  for the kernel  $K \subset G$  of cardinality  $|D_K| = \Delta[K]$ .

Fix any subset  $B \subset G$  such that  $|B| = |D_A|$  and  $|h^{-1}(x) \cap B| = 1$  for any  $x \in D_A$ . We claim that the set  $C = BD_K$  is a difference basis for  $h^{-1}(A)$ .

Since  $D_A$  is a difference basis for the set  $A$ , for any  $a \in h^{-1}(A)$  there are elements  $a_1, a_2 \in D_A$  such that  $h(a) = a_1 a_2^{-1}$ . Then  $a = b_1 b_2^{-1} k$  for some  $b_1, b_2 \in B, k \in K$ . The normality of the subgroup  $K$  in  $G$  implies that  $a = b_1 b_2^{-1} k = b_1 k' b_2^{-1}$  for some  $k' \in K$ . Taking into account that  $D_K$  is a difference basis for the kernel  $K$ , find elements  $k_1, k_2 \in D_K$  such that  $k' = k_1 k_2^{-1}$ . Then

$$a = b_1 b_2^{-1} k = b_1 k' b_2^{-1} = b_1 k_1 k_2^{-1} b_2^{-1} = (b_1 k_1)(b_2 k_2)^{-1} \in CC^{-1}$$

and

$$\Delta[h^{-1}(A)] \leq |C| \leq |D_A| \cdot |D_K| = \Delta[A] \cdot \Delta[K].$$

□

#### 5. DIFFERENCE BASES IN RINGS

In this section we construct difference bases for subsets of finite rings. All rings considered in this section have the unit. For a ring  $R$  by  $U(R)$  we denote the multiplicative group of invertible elements in  $R$ . An element  $x$  of a ring  $R$  is called *invertible* if there exists an element  $x^{-1} \in R$  such that  $xx^{-1} = x^{-1}x = 1$ . The group  $U(R)$  is called *the group of units* of the ring  $R$ .

The *characteristic* of a finite ring  $R$  is the smallest natural number  $n$  such that  $nx = 0$  for every  $x \in R$ . A non-empty subset  $I$  of a ring  $R$  is called an *ideal* in  $R$  if  $I \neq R$ ,  $I - I \subset I$  and  $IR \cup RI \subset I$ . The spectrum  $\text{Spec}(R)$  of a ring is the set of all maximal ideals of  $R$ . For any maximal ideal  $I$  of a commutative ring  $R$  the quotient ring  $R/I$  is a field.

A ring  $R$  is called *local* if it contains a unique maximal ideal, which is denoted by  $I_m$ . The quotient ring  $R/I_m$  is a field called the *residue field* of the local ring  $R$ .

By [3, Ch.6], for every prime number  $p$  and natural numbers  $k, r$  there exists a unique local ring  $\text{GR}(p^k, r)$  called the *Galois ring* of characteristic  $p^k$  whose additive group is isomorphic to the group  $(C_{p^k})^r$ , the maximal ideal  $I_m$  coincides with the principal ideal  $pR$  generated by  $p \cdot 1$  and whose residue field  $\text{GR}(p^k, r)$  contains  $p^r$  elements. The Galois ring  $\text{GR}(p^k, r)$  can be constructed as the quotient ring  $\mathbb{Z}[x]/(p^k, f(x))$  of the ring  $\mathbb{Z}[x]$  of polynomials with integer coefficients by the ideal  $(p^k, f(x))$  generated by the constant  $p^k$  and a carefully selected monic polynomial  $f \in \mathbb{Z}[x]$  of degree  $r$ , which is irreducible over the field  $\mathbb{Z}/p\mathbb{Z}$ , see [3, 6.1]. For  $k = 1$  the Galois ring  $\text{GR}(p^k, r)$  is a field, and for  $r = 1$  the Galois ring  $\text{GR}(p^k, r)$  is isomorphic to the ring  $\mathbb{Z}/p^k\mathbb{Z}$ .

The following description of the multiplicative group of a Galois ring is taken from Theorem 6.1.7 of the book [3].

**Theorem 5.1.** *Let  $p$  be a prime number and  $k, r$  be natural numbers. The multiplicative group  $U(R)$  of a Galois ring  $R := \text{GR}(p^k, r)$  is isomorphic to:*

$$\begin{cases} C_{p^{r-1}} \times C_{p^{k-1}}^{r-1} & \text{if either } p \text{ is odd or } p = 2 \text{ and } k \leq 2, \\ C_{2^{r-1}} \times C_2 \times C_{2^{k-2}} \times C_{2^{k-1}}^{r-1} & \text{if } p = 2 \text{ and } k \geq 3. \end{cases}$$

The following theorem is our principal tool for evaluating the difference sizes of finite Abelian groups of odd order.

**Theorem 5.2.** *Let  $R$  be a finite commutative ring  $R$  with unit and  $(1+1) \in U(R)$  and let  $h : G \rightarrow R \times R$  be a surjective homomorphism from a group  $G$  onto the Abelian group of the ring  $R \times R$ . Let  $K = h^{-1}(0, 0)$  be the kernel of the homomorphism  $h$ . Then*

$$\Delta[G] \leq \Delta[K] \cdot |R| - |\text{Spec}(R)| + \sum_{I \in \text{Spec}(R)} \Delta[h^{-1}(I \times R)].$$

If the ring  $R$  is local, then  $\Delta[G] \leq \Delta[K] \cdot |R| - 1 + \Delta[h^{-1}(I_{\mathfrak{m}} \times R)]$ .

*Proof.* First we observe that  $R = U(R) \cup \bigcup_{I \in \text{Spec}(R)} I$ . Indeed, if an element  $x \in R$  is not invertible, then the set  $xR = \{xy : y \in R\}$  is an ideal in  $R$ , contained in some maximal ideal  $I \in \text{Spec}(R)$ . This implies that  $R = U(R) \cup \bigcup_{I \in \text{Spec}(R)} I$  and hence  $G = h^{-1}(U(R) \times R) \cup \bigcup_{I \in \text{Spec}(R)} h^{-1}(I \times R)$ .

**Lemma 5.3.** *The set  $B = \{(x, x^2) : x \in R\}$  is a difference basis for the set  $U(R) \times R$  in the additive group  $R \times R$ .*

*Proof.* Given any pair  $(a, b) \in U(R) \times R$ , we should find two elements  $x, y \in R$  such that  $(a, b) = (x - y, x^2 - y^2)$ . Solving this system of equations in the ring  $R$ , we get the solution

$$\begin{cases} x = 2^{-1}(ba^{-1} + a) \\ y = 2^{-1}(ba^{-1} - a). \end{cases}$$

□

By Lemma 5.3, the set  $B = \{(x, x^2) : x \in R\}$  is a difference basis for the set  $U(R) \times R$  in  $R \times R$ . So,  $\Delta[U(R) \times R] \leq |B| = |R|$ . By Proposition 4.2,

$$h^{-1}(U(R) \times R) \leq \Delta[K] \cdot \Delta[U(R) \times R] \leq \Delta[K] \cdot |R|$$

and by Proposition 4.1,

$$\Delta[G] \leq \Delta[h^{-1}(U(R) \times R)] + \sum_{I \in \text{Spec}(R)} (\Delta[h^{-1}(I \times R)] - 1) \leq \Delta[K] \cdot |R| - |\text{Spec}(R)| + \sum_{I \in \text{Spec}(R)} \Delta[h^{-1}(I \times R)].$$

□

Our next theorem will be applied for evaluating the difference characteristics of Abelian 2-groups. This theorem exploits the structure of a (non)associative ring. By a *(non)associative ring* we understand an Abelian group  $R$  endowed with a binary operation  $\circ : R \times R \rightarrow R$  which is distributive in the sense that  $x \circ (y + z) = x \circ y + x \circ z$  and  $(x + y) \circ z = x \circ z + y \circ z$  for all  $x, y, z \in R$ . A (non)associative ring  $R$  is called a *ring* if its binary operation  $\circ$  is associative. In the opposite case  $R$  is called a *non-associative ring*.

For any (non)associative ring  $R$  the product  $R \times R$ , endowed with the binary operation

$$(x, y) \star (x', y') = (x + x', y + y' + x \circ x'),$$

is a group. The inverse element to  $(x, y)$  in this group is  $(-x, -y + x \circ x)$ . The product  $R \times R$  endowed with this group operation will be denoted by  $R \star R$ . The group  $R \star R$  is commutative if and only if the binary operation  $\circ$  on  $R$  is commutative. For a (non)associative ring  $R$  let  $U(R)$  be the set of all elements  $a \in R$  such that the map  $R \rightarrow R, x \mapsto x \circ a$ , is bijective. The following theorem was known for semifields, see [9, 4.1].

**Theorem 5.4.** *For any finite (non)associative ring  $R$  the set  $B = \{(x, x \circ x) : x \in R\}$  is a difference basis for the set  $U(R) \times R$  in the group  $R \star R$ .*

*Proof.* Given any pair  $(a, b) \in U(R) \times R$ , we need to find elements  $x, y \in R$  such that  $(a, b) = (x, x \circ x) \star (y, y \circ y)^{-1}$ . The definition of the group operation  $\star$  implies that  $(y, y \circ y)^{-1} = (-y, 0)$ . Then the equality  $(a, b) = (x, x \circ x) \star (y, y \circ y)^{-1}$  turns into  $(a, b) = (x, x \circ x) \star (-y, 0) = (x - y, x \circ x - x \circ y) = (x - y, x \circ (x - y))$ . Since  $a \in U(R)$ , there exists an element  $x \in R$  such that  $x \circ a = b$ . Let  $y = x - a$  and observe that the pair  $(x, y)$  has the required property:

$$(x, x \circ x) \star (y, y \circ y)^{-1} = (x - y, x \circ (x - y)) = (a, x \circ a) = (a, b).$$

□

Theorem 5.4 suggests the problem of detecting the structure of the group  $R \star R$  for various rings  $R$ . For Galois rings  $GR(p^k, r)$  this problem is answered in the following two theorems.

**Theorem 5.5.** *Let  $p$  be a prime number, and  $k, r$  be natural numbers. For the Galois ring  $R := GR(p^k, r)$  the group  $R \star R$  is isomorphic to*

$$\begin{cases} C_{p^k}^r \times C_{p^k}^r & \text{if } p \geq 3, \\ C_{2^{k+1}}^r \times C_{2^{k-1}}^r & \text{if } p = 2. \end{cases}$$

*Proof.* First observe that the commutativity of the Galois ring  $R$  implies the commutativity of the group  $R \star R$ . To determine the structure of the group  $R \star R$  we shall calculate the orders of its elements. Let us recall that the *order* of an element  $x$  in an Abelian group  $G$  is the smallest number  $n \in \mathbb{N}$  such that  $nx = 0$ .

Let us fix an element  $(x, y) \in R \times R$  and evaluate its order in the group  $R \star R$ . By induction it can be shown that for every  $s \in \mathbb{N}$  we get  $(x, y)^s = (sx, sy + \frac{s(s-1)}{2}x^2)$ .

If  $p \geq 3$ , then  $(x, y)^{p^k} = (p^k x, p^k y + p^k \frac{p^k-1}{2}x^2) = (0, 0)$  as  $p^k \cdot z = 0$  for each element  $z \in R$ . On the other hand,  $(x, y)^{p^{k-1}} = (0, 0)$  if and only if  $p^{k-1}x = p^{k-1}y = 0$  if and only if  $(x, y) \in pR \times pR$ , which implies that the set of elements of order  $p^{k-1}$  has cardinality  $p^{2(k-1)r}$ . It remains to observe that up to an isomorphism,  $C_{p^k}^{2r}$  is the unique Abelian  $p$ -group of cardinality  $p^{2kr}$  that contains  $p^{2kr}$  elements of order  $\leq p^k$  and  $p^{2kr} - p^{2(k-1)r}$  elements of order  $p^k$ .

Next, assume that  $p = 2$ . In this case  $(x, y)^{2^{k+1}} = (2^{k+1}x, 2^{k+1}y + 2^k(2^{k+1} - 1)x^2) = (0, 0)$ , which means that each element of the group  $R \star R$  has order  $\leq 2^{k+1}$ . Observe that  $(x, y)^{2^k} = (2^k x, 2^k y + 2^{k-1}(2^k - 1)x^2) = (0, 2^{k-1}(2^k - 1)x^2)$ , which implies that  $(x, y)^{2^k} \neq (0, 0)$  if and only if  $x^2 \notin I_m = 2R$  if and only if  $x \in U(R)$ . This means that the 2-group  $R \star R$  has exactly  $|U(R) \times R| = (2^{kr} - 2^{(k-1)r})2^{kr} = 2^{(2k-1)r}(2^r - 1)$  elements of order  $2^{k+1}$ .

Next, for any  $i \in \{0, \dots, k\}$ , we calculate the number of elements of order  $> 2^{k-i}$  in  $R \star R$ . Observe that an element  $(x, y) \in R \star R$  has order  $> 2^{k-i}$  if and only if  $(2^{k-i}x, 2^{k-i}y + 2^{k-i-1}(2^{k-i} - 1)x^2) \neq (0, 0)$  if and only if either  $x \notin 2^i R$  or  $x \in 2^i R$  and  $y \notin 2^i R$ . So, the set of elements of order  $> 2^{k-i}$  coincides with  $((R \setminus 2^i R) \times R) \cup (2^i R \times (R \setminus 2^i R))$  and hence has cardinality

$$|R \setminus 2^i R| \cdot |R| + |2^i R| \cdot |R \setminus 2^i R| = (|R| - |2^i R|) \cdot (|R| + |2^i R|) = |R|^2 - |2^i R|^2 = 2^{2kr} - 2^{2(k-i)r} = 2^{2(k-i)r}(2^{2ir} - 1).$$

This information is sufficient to detect the isomorphic type of the group  $R \star R$ . By [11, 4.2.6], the Abelian 2-group  $R \star R$  is isomorphic to the product  $H = \prod_{i=1}^{k+1} C_{2^i}^{m_i}$  for some numbers  $m_1, \dots, m_{k+1} \in \{0\} \cup \mathbb{N}$ . Observe that the group  $H$  contains  $(2^{(k+1)m_{k+1}} - 2^{km_{k+1}}) \cdot \prod_{i=1}^k 2^{im_i} = 2^{km_{k+1}}(2^{m_{k+1}} - 1) \cdot \prod_{i=1}^k 2^{im_i}$  elements of order  $2^{k+1}$ . Taking into account that the group  $R \star R$  contains  $2^{(2k-1)r}(2^r - 1)$  elements of order  $2^{k+1}$ , we conclude that  $m_{k+1} = r$ .

Next, observe that the group  $H$  contains exactly

$$\begin{aligned} & |C_{2^{k+1}}^{m_{k+1}} \times C_{2^k}^{m_k} - C_{2^{k-1}}^{m_{k+1}} \times C_{2^{k-1}}^{m_k}| \cdot \prod_{i=1}^{k-1} |C_{2^i}^{m_i}| = \\ & = (2^{(k+1)r+km_k} - 2^{(k-1)(r+m_k)}) \cdot \prod_{i=1}^{k-1} 2^{im_i} = 2^{(k-1)(r+m_k)}(2^{2r+m_k} - 1) \cdot \prod_{i=1}^{k-1} 2^{im_i} \end{aligned}$$

elements of order  $\geq 2^k$ . Taking into account that the group  $R \star R$  contains exactly  $2^{2(k-1)n}(2^{2r} - 1)$  elements of order  $\geq 2^k$ , we conclude that  $m_k = 0$ .

The group  $H$  contains exactly

$$\begin{aligned} |C_{2^{k+1}}^{m_{k+1}} \times C_{2^k}^{m_k} \times C_{2^{k-1}}^{m_{k-1}} - C_{2^{k-2}}^{m_{k-1}} \times C_{2^{k-2}}^{m_k} \times C_{2^{k-2}}^{m_{k-1}}| \cdot \prod_{i=1}^{k-2} |C_{2^i}^{m_i}| = \\ = (2^{(k+1)r+(k-1)m_{k-1}} - 2^{(k-2)(r+m_{k-1})}) \cdot \prod_{i=1}^{k-2} 2^{im_i} = 2^{(k-2)(r+m_{k-1})} (2^{3r+m_{k-1}} - 1) \cdot \prod_{i=1}^{k-2} 2^{im_i} \end{aligned}$$

elements of order  $\geq 2^{k-1}$ . Taking into account that the group  $R \star R$  contains exactly  $2^{(k-2)r} (2^{4r} - 1)$  elements of order  $\geq 2^{k-1}$ , we conclude that  $m_{k-1} = r$ .

Taking into account that  $|C_{2^{k+1}}^{m_{k+1}} \times C_{2^{k-1}}^{m_{k-1}}| = |C_{2^{k+1}}^r \times C_{2^{k-1}}^r| = 2^{2kr} = |R \star R|$ , we conclude that  $m_i = 0$  for  $i < k-1$  and hence the group  $R \star R$  is isomorphic to  $C_{2^{k+1}}^r \times C_{2^{k-1}}^r$ .  $\square$

## 6. EVALUATING THE DIFFERENCE CHARACTERISTICS OF ABELIAN $p$ -GROUPS

In this section we shall evaluate the difference characteristics of finite Abelian  $p$ -groups for an odd prime number  $p$ . We recall that a group  $G$  is called a  $p$ -group if each element  $x \in G$  generates a finite cyclic group of order  $p^k$  for some  $k \in \mathbb{N}$ . A finite group  $G$  is a  $p$ -group if and only if  $|G| = p^k$  for some  $k \in \mathbb{N}$ .

It is well-known that each Abelian  $p$ -group  $G$  is isomorphic to the product  $\prod_{i=1}^r C_{p^{k_i}}$  of cyclic  $p$ -groups. The number  $r$  of cyclic groups in this decomposition is denoted by  $r(G)$  and called the *rank* of  $G$ .

Applying Theorem 5.2 to the Galois ring  $R := \text{GR}(p^k, r)$  and taking into account that its additive group is isomorphic to  $C_{p^k}^r$  and  $pR$  coincides with the maximal ideal of  $R$ , we get the following corollary.

**Corollary 6.1.** *Let  $p$  be an odd prime number,  $k, r$  be natural numbers. Let  $h : G \rightarrow C_{p^k}^{2r}$  be a surjective homomorphism and  $K$  be its kernel. Then*

$$\Delta[G] \leq \Delta[K] \cdot p^{kr} + \Delta[h^{-1}(C_{p^{k-1}}^r \times C_{p^k}^r)] - 1$$

and

$$\eth[G] \leq \eth[K] + \frac{1}{\sqrt{p^r}} \cdot \eth[h^{-1}(C_{p^{k-1}}^r \times C_{p^k}^r)] - \frac{1}{\sqrt{|G|}}.$$

This corollary implies the following recursive upper bound for difference characteristics of finite Abelian  $p$ -groups.

**Theorem 6.2.** *Let  $p$  be an odd prime number,  $k_1, \dots, k_m$  be natural numbers, and  $k, r$  be natural numbers such that  $2r \leq m$  and  $k \leq \min_{1 \leq i \leq 2r} k_i$ . Then*

$$\Delta\left[\prod_{i=1}^m C_{p^{k_i}}\right] \leq \Delta\left[\prod_{i=1}^{2r} C_{p^{k_i-k}} \times \prod_{i=2r+1}^m C_{p^{k_i}}\right] \cdot p^{kr} + \Delta\left[\prod_{i=1}^r C_{p^{k_{i-1}}} \times \prod_{i=r+1}^m C_{p^{k_i}}\right] - 1$$

and

$$\eth\left[\prod_{i=1}^m C_{p^{k_i}}\right] \leq \eth\left[\prod_{i=1}^{2r} C_{p^{k_i-k}} \times \prod_{i=2r+1}^m C_{p^{k_i}}\right] + \frac{1}{\sqrt{p^r}} \cdot \eth\left[\prod_{i=1}^r C_{p^{k_{i-1}}} \times \prod_{i=r+1}^m C_{p^{k_i}}\right] - \prod_{i=1}^m \frac{1}{\sqrt{p^{k_i}}}.$$

The recursive formulas from the preceding theorem will be used in the following upper bound for the difference characteristic of finite Abelian  $p$ -group.

**Theorem 6.3.** *For any prime number  $p \geq 11$ , any finite Abelian  $p$ -group  $G$  has difference characteristic*

$$\eth[G] \leq \frac{\sqrt{p}-1}{\sqrt{p}-3} \cdot \sup_{k \in \mathbb{N}} \eth[C_{p^k}] \leq \frac{\sqrt{p}-1}{\sqrt{p}-3} \cdot \frac{24}{\sqrt{293}}.$$

*Proof.* For a prime number  $p$  and a natural number  $r$  let  $\text{Ab}_p^r$  be the class of Abelian  $p$ -groups of rank  $r$ . Let also  $\text{Ab}_p^{<r} = \bigcup_{n < r} \text{Ab}_p^n$  and  $\text{Ab}_p^{\leq r} = \bigcup_{n \leq r} \text{Ab}_p^n$ . Let  $\text{Ab}_p := \bigcup_{r \in \mathbb{N}} \text{Ab}_p^r$  be the family of finite Abelian  $p$ -groups. For a class  $\mathcal{C}$  of finite groups we put  $\eth[\mathcal{C}] := \sup_{G \in \mathcal{C}} \eth[G]$ . By Theorem 2.1,  $\eth[\mathcal{C}] \leq \frac{4}{\sqrt{3}}$ .

**Lemma 6.4.** *For any odd prime number  $p$  and any natural number  $r$  we get the upper bound*

$$\eth[\text{Ab}_p^r] \leq \eth[\text{Ab}_p^{<r}] + \frac{1}{\sqrt{p^{\lfloor r/2 \rfloor}}} \cdot \eth[\text{Ab}_p^{\leq r}] \leq \eth[\text{Ab}_p^{<r}] + \frac{1}{\sqrt{p^{\lfloor r/2 \rfloor}}} \cdot \eth[\text{Ab}_p].$$

Consequently,

$$\delta[\text{Ab}_p^r] \leq \delta[\text{Ab}_p^1] + \delta[\text{Ab}_p] \cdot \sum_{i=2}^r \frac{1}{\sqrt{p^{[i/2]}}} \leq \delta[\text{Ab}_p^1] + \delta[\text{Ab}_p] \cdot \sum_{i=1}^{\lfloor r/2 \rfloor} \frac{2}{\sqrt{p^i}}$$

*Proof.* Any group  $G \in \text{Ab}^r$  is isomorphic to the product  $\prod_{i=1}^r C_{p^{k_i}}$  for a unique non-decreasing sequence  $(k_i)_{i=1}^r$  of natural numbers. Let  $k = k_1$  and  $m = \lfloor \frac{r}{2} \rfloor$ . Since  $k_1 - k = 0$ , the group  $\prod_{i=1}^{2m} C_{p^{k_i-k}} \times \prod_{i=2m+1}^r C_{p^{k_i}}$  has rank  $< r$ . Applying Theorem 6.2, we conclude that

$$\begin{aligned} \delta[G] &= \delta\left[\prod_{i=1}^r C_{p^{k_i}}\right] < \delta\left[\prod_{i=1}^{2m} C_{p^{k_i-k}} \times \prod_{i=2m+1}^r C_{p^{k_i}}\right] + \frac{1}{\sqrt{p^m}} \cdot \delta\left[\prod_{i=1}^m C_{p^{k_i-1}} \times \prod_{i=m+1}^r C_{p^{k_i}}\right] \leq \\ &\leq \delta[\text{Ab}_p^{<r}] + \frac{1}{\sqrt{p^m}} \cdot \delta[\text{Ab}_p^{\leq r}] \leq \delta[\text{Ab}_p^{<r}] + \frac{1}{\sqrt{p^m}} \cdot \delta[\text{Ab}_p]. \end{aligned}$$

□

Lemma 6.4 implies that

$$\delta[\text{Ab}_p] \leq \delta[\text{Ab}_p^1] + \delta[\text{Ab}_p] \cdot \sum_{i=1}^{\infty} \frac{2}{\sqrt{p^i}} = \delta[\text{Ab}_p^1] + \delta[\text{Ab}_p] \cdot \frac{2}{\sqrt{p}-1}$$

and after transformations

$$\delta[\text{Ab}_p] \leq \delta[\text{Ab}_p^1] \cdot \left(1 - \frac{2}{\sqrt{p}-1}\right)^{-1} = \delta[\text{Ab}_p^1] \cdot \frac{\sqrt{p}-1}{\sqrt{p}-3} = \frac{\sqrt{p}-1}{\sqrt{p}-3} \cdot \sup_{k \in \mathbb{N}} \delta[C_{p^k}] \leq \frac{\sqrt{p}-1}{\sqrt{p}-3} \cdot \frac{24}{\sqrt{293}}.$$

In the last inequality we use the upper bound  $\sup_{k \in \mathbb{N}} \delta[C_{p^k}] \leq \frac{24}{\sqrt{293}}$  from Theorem 2.5. □

Theorem 6.2 implies:

**Corollary 6.5.** *For any odd prime number  $p$  and natural numbers  $k, n$  the groups  $C_{p^k}^{2n}$  and  $C_{p^k}^{2n+1}$  have difference characteristics*

$$\delta[C_{p^k}^{2n}] \leq 1 - \frac{1}{p^{kr}} + \frac{1}{\sqrt{p^r}} \cdot \delta[C_{p^{k-1}}^n \times C_{p^k}^n] < 1 + \frac{1}{\sqrt{p^r}} \cdot \delta[\text{Ab}_p]$$

and

$$\delta[C_{p^k}^{2n+1}] \leq \delta[C_{p^k}^{2n}] - \frac{1}{p^{kr}} + \frac{1}{\sqrt{p^r}} \cdot \delta[C_{p^{k-1}}^n \times C_{p^k}^{n+1}] < \delta[C_{p^k}^{2n}] + \frac{1}{\sqrt{p^r}} \cdot \delta[\text{Ab}_p].$$

## 7. EVALUATING THE DIFFERENCE CHARACTERISTICS OF 2-GROUPS

In this section we elaborate tools for evaluating the difference characteristics of 2-groups. The following corollary is a counterpart of Corollary 6.1.

**Corollary 7.1.** *Let  $k, r$  be natural numbers. Let  $h : G \rightarrow C_{2^{k+1}}^r \times C_{2^{k-1}}^r$  be a surjective homomorphism and  $K$  be its kernel. Then*

$$\Delta[G] \leq \Delta[K] \cdot 2^{kr} + \Delta[h^{-1}(C_{2^k}^r \times C_{2^{k-1}}^r)] - 1$$

and

$$\delta[G] \leq \delta[K] + \frac{1}{\sqrt{2^r}} \cdot \delta[h^{-1}(C_{2^k}^r \times C_{2^{k-1}}^r)] - \frac{1}{\sqrt{|G|}}.$$

*Proof.* Consider the Galois ring  $R := \text{GR}(2^k, r)$ , whose additive group is isomorphic to  $C_{2^r}^k$ . Its maximal ideal  $I_m$  coincides with the subgroup  $2R$  of  $R$ , which consists of elements of order  $\leq 2^{k-1}$  in  $R$ . The subset  $pR \times R$  of the group  $R \star R$  has cardinality  $2^{(2k-1)r}$  and consists of elements of order  $\leq 2^k$  in  $R \star R$ . By Theorem 5.5, the group  $R \star R$  is isomorphic to  $C_{2^{k+1}}^r \times C_{2^{k-1}}^r$ . It is easy to see that  $C_{2^k}^r \times C_{2^{k-1}}^r$  is the unique subgroup of cardinality  $2^{(2k-1)r}$  consisting of elements of order  $\leq 2^k$ . Therefore, the group  $C_{2^{k+1}}^r \times C_{2^{k-1}}^r$  can be identified with the group  $R \star R$  and its subgroup  $C_{2^k}^r \times C_{2^{k-1}}^r$  with the subgroup  $2R \times R$  of the group  $R \times R$ . By Theorem 5.4, the set  $B = \{(x, x^2) : x \in R\}$  is a difference base for the set  $U(R) \times R$  in the group  $R \times R$ . So,  $\Delta[U(R) \times R] \leq |R| = 2^k$ . By Propositions 4.1 and 4.2,

$$\Delta[G] \leq \Delta[K] \cdot \Delta[U(R) \times R] - 1 + \Delta[h^{-1}(I_m \times R)] \leq \Delta[K] \cdot |R| - 1 + \Delta[h^{-1}(2R \times R)]$$



and hence

$$\delta[G] = \frac{\Delta[K] \cdot |R|}{\sqrt{|K| \cdot |R|^2}} - \frac{1}{\sqrt{|G|}} + \frac{\Delta[h^{-1}(2R \times R)]}{\sqrt{|K| \cdot |2R| \cdot |R| \cdot |R/2R|}} = \delta[K] - \frac{1}{\sqrt{|G|}} + \frac{1}{\sqrt{2^r}} \cdot \delta[h^{-1}(2R \times R)].$$

□

This corollary implies the following recursive upper bound for difference characteristics of finite Abelian 2-groups.

**Theorem 7.2.** *Let  $k_1, \dots, k_m$  be natural numbers, and  $k, r$  be natural numbers such that  $2r \leq m$ ,  $k+1 \leq \min_{1 \leq i \leq r} k_i$  and  $k-1 \leq \min_{r < i \leq 2r} k_i$ . Then*

$$\Delta\left[\prod_{i=1}^m C_{2^{k_i}}\right] \leq \Delta\left[\prod_{i=1}^r C_{2^{k_i-k-1}} \times \prod_{i=r+1}^{2r} C_{2^{k_i-k+1}} \times \prod_{i=2r+1}^m C_{2^{k_i}}\right] \cdot 2^{kr} + \Delta\left[\prod_{i=1}^r C_{2^{k_i-1}} \times \prod_{i=r+1}^m C_{2^{k_i}}\right] - 1$$

and

$$\delta\left[\prod_{i=1}^m C_{2^{k_i}}\right] \leq \delta\left[\prod_{i=1}^r C_{2^{k_i-k-1}} \times \prod_{i=r+1}^{2r} C_{2^{k_i-k+1}} \times \prod_{i=2r+1}^m C_{2^{k_i}}\right] + \frac{1}{\sqrt{2^r}} \cdot \delta\left[\prod_{i=1}^r C_{2^{k_i-1}} \times \prod_{i=r+1}^m C_{2^{k_i}}\right] - \prod_{i=1}^m \frac{1}{2^{k_i}}.$$

Now we shall evaluate the difference characteristics of the 2-groups  $C_{2^n}^r$ .

**Proposition 7.3.** *For any  $n \in \mathbb{N}$  the groups  $C_2^{2n}$  and  $C_2^{2n+1}$  have difference sizes*

$$\frac{1 + \sqrt{2^{2n+3} - 7}}{2} \leq \Delta[C_2^{2n}] < 2^{n+1} \quad \text{and} \quad \frac{1 + \sqrt{2^{2n+4} - 7}}{2} \leq \Delta[C_2^{2n+1}] < 3 \cdot 2^n$$

and difference characteristics

$$\sqrt{2} < \frac{1 + \sqrt{2^{2n+3} - 7}}{2^{n+1}} \leq \delta[C_2^{2n}] < 2 \quad \text{and} \quad \sqrt{2} < \frac{1 + \sqrt{2^{2n+4} - 7}}{\sqrt{2} \cdot 2^{n+1}} \leq \delta[C_2^{2n+1}] < \frac{3}{\sqrt{2}}.$$

*Proof.* The lower bound  $\frac{1 + \sqrt{8|G| - 7}}{2} \leq \Delta[G]$  follows from Theorem 3.1.

The upper bound will be derived from Proposition 2.2(3) which implies that

$$\Delta[C_2^{2n}] < |C_2^n| + |C_2^m| = 2 \cdot 2^n$$

and

$$\Delta[C_2^{2n+1}] < |C_2^n| + |C_2^{n+1}| = 3 \cdot 2^n.$$

These upper bounds imply

$$\delta[C_2^{2n}] < 2 \quad \text{and} \quad \delta[C_2^{2n+1}] < \frac{3}{\sqrt{2}}.$$

□

**Proposition 7.4.** *For any  $n \in \mathbb{N}$  the group  $C_4^n$  has the difference characteristic*

$$\delta[C_4^n] \leq 1 + \frac{1}{\sqrt{2^n}} \cdot \delta[C_2^n] - \frac{1}{2^n} < 1 - \frac{1}{2^n} + \frac{3}{\sqrt{2^{n+1}}}.$$

*Proof.* Consider the numbers  $k_1 = \dots = k_n = 2$  and  $k_{n+1} = \dots = k_{2n} = 1$ . By Theorem 7.2,

$$\delta[C_4^n] = \delta\left[\prod_{i=1}^{2n} C_{2^{k_i}}\right] \leq \delta[C_1] - \frac{1}{2^n} + \frac{1}{\sqrt{2^n}} \delta\left[\prod_{i=1}^n C_{2^{k_i-1}} \times \prod_{i=n+1}^{2n} C_{2^{k_i}}\right] = 1 - \frac{1}{2^n} + \frac{1}{\sqrt{2^n}} \delta[C_2^n] < 1 - \frac{1}{2^n} + \frac{3}{\sqrt{2^{n+1}}}.$$

In the last inequality we used the upper bound  $\delta[C_2^n] < \frac{3}{\sqrt{2}}$ , proved in Proposition 7.3. □

Theorem 7.2, Proposition 7.4 and Theorem 2.1 imply:

**Corollary 7.5.** *For any  $k, n \in \mathbb{N}$  the group  $C_{2^k}^{2n}$  and  $C_{2^k}^{2n+1}$  have the difference characteristics*

$$\delta[C_{2^k}^{2n}] < \delta[C_4^n] - \frac{1}{2^{kn}} + \frac{1}{\sqrt{2^n}} \delta[C_{2^{k-1}}^n \times C_{2^k}^n] < 1 + \frac{1}{\sqrt{2^n}} \left( \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{3}} \right)$$

and

$$\delta[C_{2^k}^{2n+1}] < \delta[C_4^n \times C_{2^k}] + \frac{1}{\sqrt{2^n}} \delta[C_{2^{k-1}}^n \times C_{2^k}^{n+1}] < \delta[C_{2^k}] \cdot \left( 1 + \frac{3}{\sqrt{2^{n+1}}} \right) + \frac{4}{\sqrt{3 \cdot 2^n}}.$$

8. THE DIFFERENCE CHARACTERISTIC OF THE GROUPS  $R \times U(R)$ 

In this section we obtain an upper bound for the difference characteristics of the groups  $R \times U(R)$ , which are products of the additive group of a ring  $R$  and the multiplicative group  $U(R)$  of its units.

**Theorem 8.1.** *For any finite ring  $R$  and its multiplicative group  $U(R)$  of units the set  $B = \{(x, x) : x \in U(R)\}$  is a difference base for the set  $A = \{(x, y) \in R \times U(R) : \exists z \in U(R) \ (y - 1)z = x\}$  in the group  $R \times U(R)$ . If the ring  $R$  is local with maximal ideal  $I_m$  and the residue field  $F = R/I_m$ , then*

$$\Delta[R \times U(R)] = |U(R)| - 2 + \Delta[I_m \times U(R)] + \Delta[R \times (1 + I_m)] \leq |U(R)| - 2 + \frac{4}{\sqrt{3}} \frac{|R|}{|F|} (\sqrt{|F| - 1} + \sqrt{|F|}).$$

and

$$\delta[R \times U(R)] < \sqrt{1 - \frac{1}{|F|}} + \frac{4}{\sqrt{3}} \left( \frac{1}{\sqrt{|F|}} + \frac{1}{\sqrt{|F| - 1}} \right).$$

*Proof.* Given a pair  $(x, y) \in A$ , we should find two elements  $a, b \in U(R)$  such that  $(a - b, ab^{-1}) = (x, y)$ . Since  $(x, y) \in A$ , there exists  $z \in U(R)$  such that  $(y - 1)z = x$ . Then for the pair  $(a, b) := (yz, z)$  we get the required equality  $(a - b, ab^{-1}) = (yz - z, yzz^{-1}) = ((y - 1)z, y) = (x, y)$ .

Now assume that the ring  $R$  is local and consider its (unique) maximal ideal  $I_m$ . Let  $\pi : R \rightarrow R/I_m$  be the homomorphism of  $R$  onto its residue field  $F := R/I_m$ . It follows that  $|R| = |F| \cdot |I_m|$ . The maximality of the ideal  $I_m$  guarantees that  $U(R) = R \setminus I_m$  and  $1 + I_m = \pi^{-1}(\pi(1))$  is a multiplicative subgroup of  $U(R)$ . We claim that  $(I_m \times U(R)) \cup (R \times (1 + I_m)) \cup A = R \times U(R)$ . Indeed, if a pair  $(x, y) \in R \times U(R)$  does not belong to  $(I_m \times U(R)) \cup (R \times (1 + I_m))$ , then  $x \in U(R)$  and  $y \notin 1 + I_m$ . It follows that  $1 - y \notin I_m$  and hence the element  $1 - y$  is invertible, so we can find  $z = (1 - y)^{-1}x \in U(R)$  and conclude that  $(x, y) \in A$ . Theorem 2.1 and the subadditivity of the difference size proved in Proposition 2.2(3) guarantee that

$$\begin{aligned} \Delta[R \times U(R)] &\leq \Delta[A] + \Delta[I_m \times U(R)] + \Delta[R \times (1 + I_m)] - 2 \leq \\ &\leq |U(R)| - 2 + \frac{4}{\sqrt{3}} (\sqrt{|I_m| \times |U(R)|} + \sqrt{|R| \times |I_m|}) = |U(R)| - 2 + \frac{4}{\sqrt{3}} \sqrt{|I_m|} (\sqrt{|R|} - \sqrt{|I_m|} + \sqrt{|R|}) = \\ &= |U(R)| - 2 + \frac{4}{\sqrt{3}} |I_m| (\sqrt{|R/I_m|} - 1 + \sqrt{|R/I_m|}) \leq |U(R)| - 2 + \frac{4}{\sqrt{3}} \frac{|R|}{|F|} (\sqrt{|F| - 1} + \sqrt{|F|}). \end{aligned}$$

Dividing  $\Delta[R \times U(R)]$  by  $\sqrt{|R \times U(R)|} = \sqrt{|R|(|R| - |I_m|)} = |R| \sqrt{1 - \frac{1}{|F|}}$ , we get the required upper bound for the difference characteristic

$$\delta[R \times U(R)] < \sqrt{\frac{|U(R)|}{|R|}} + \frac{4}{\sqrt{3}} \frac{1}{\sqrt{|F|^2 - |F|}} (\sqrt{|F| - 1} + \sqrt{|F|}) = \sqrt{1 - \frac{1}{|F|}} + \frac{4}{\sqrt{3}} \left( \frac{1}{\sqrt{|F|}} + \frac{1}{\sqrt{|F| - 1}} \right).$$

□

Combining Theorem 8.1 with Theorem 5.1 describing the structure of the multiplicative groups of the Galois rings  $GR(p^k, r)$ , we get the following two corollaries.

**Corollary 8.2.** *Let  $p$  be a prime number and  $k, r$  be natural numbers such that either  $p \geq 3$  or  $p = 2$  and  $k \leq 2$ . The group  $G = C_{p^k}^r \times C_{p^{k-1}}^r \times C_{p^{r-1}}$  has difference characteristic*

$$\delta[G] < \sqrt{1 - \frac{1}{p^r}} + \frac{4}{\sqrt{3}} \left( \frac{1}{\sqrt{p^r}} + \frac{1}{\sqrt{p^r - 1}} \right) = 1 + O\left(\frac{1}{p^{r/2}}\right).$$

**Corollary 8.3.** *For any natural numbers  $r$  and  $k \geq 3$  the group*

$$G := C_{2^k}^r \times C_{2^{k-1}}^{r-1} \times C_{2^{k-2}} \times C_2 \times C_{2^{r-1}}$$

*has difference characteristic*

$$\delta[G] < \sqrt{1 - \frac{1}{2^r}} + \frac{4}{\sqrt{3}} \left( \frac{1}{\sqrt{2^r}} + \frac{1}{\sqrt{2^r - 1}} \right) = 1 + O\left(\frac{1}{2^{r/2}}\right).$$

## 9. THE RESULTS OF COMPUTER CALCULATIONS

In Table 3 we present the results of computer calculations of the difference sizes of all non-cyclic Abelian groups  $G$  of order  $12 \leq |G| < 96$ . In this table

$$lb[G] := \left\lceil \frac{1 + \sqrt{4|G| + 4|G_2| - 3}}{2} \right\rceil$$

is the lower bound given in Corollary 3.2.

TABLE 3. Difference sizes of non-cyclic Abelian groups  $G$  of order  $12 \leq |G| < 96$ 

$G$	$(C_2)^2 \times C_3$	$C_2 \times C_8$	$(C_4)^2$	$(C_2)^2 \times C_4$	$(C_2)^4$	$C_2 \times (C_3)^2$	$(C_2)^2 \times C_5$
$lb[G]$	5	5	5	6	6	5	6
$\Delta[G]$	5	5	6	6	6	5	6
$\partial[G]$	1,4433...	1,25	1,5	1,5	1,5	1,1785...	1,3416...
$G$	$C_2 \times C_3 \times C_4$	$(C_2)^3 \times C_3$	$(C_5)^2$	$C_3 \times C_9$	$(C_3)^3$	$(C_2)^2 \times C_7$	$C_2 \times C_{16}$
$lb[G]$	6	6	6	6	6	6	7
$\Delta[G]$	6	6	6	6	6	6	7
$\partial[G]$	1,2247...	1,2247...	1,2	1,1547...	1,1547...	1,1338...	1,2374...
$G$	$C_4 \times C_8$	$(C_2)^2 \times C_8$	$C_2 \times (C_4)^2$	$(C_2)^3 \times C_4$	$(C_2)^5$	$(C_6)^2$	$(C_2)^2 \times C_9$
$lb[G]$	7	7	7	8	9	7	7
$\Delta[G]$	7	7	8	8	10	7	7
$\partial[G]$	1,2374...	1,2374...	1,4142...	1,4142...	1,7677...	1,1666...	1,1666...
$G$	$(C_3)^2 \times C_4$	$(C_2)^3 \times C_5$	$C_2 \times C_4 \times C_5$	$(C_2)^2 \times C_{11}$	$(C_3)^2 \times C_5$	$C_2 \times C_3 \times C_8$	$C_3 \times (C_4)^2$
$lb[G]$	7	8	7	8	8	8	8
$\Delta[G]$	7	8	8	8	8	8	8
$\partial[G]$	1,1666...	1,2649...	1,2649...	1,2060...	1,1925...	1,1547...	1,1547...
$G$	$(C_2)^2 \times C_3 \times C_4$	$(C_2)^4 \times C_3$	$(C_7)^2$	$C_2 \times (C_5)^2$	$(C_2)^2 \times C_{13}$	$C_6 \times C_9$	$C_2 \times (C_3)^3$
$lb[G]$	8	9	8	8	8	8	8
$\Delta[G]$	9	10	9	8	9	9	9
$\partial[G]$	1,2990...	1,4433...	1,2857...	1,1313...	1,2480...	1,2247...	1,2247...
$G$	$C_2 \times C_4 \times C_7$	$(C_2)^3 \times C_7$	$(C_2)^2 \times C_3 \times C_5$	$(C_3)^2 \times C_7$	$C_2 \times C_{32}$	$C_4 \times C_{16}$	$C_2 \times C_4 \times C_8$
$lb[G]$	9	9	9	9	9	9	9
$\Delta[G]$	9	10	9	9	10	10	10
$\partial[G]$	1,2026...	1,3363...	1,1618...	1,1338...	1,25	1,25	1,25
$G$	$(C_2)^2 \times C_{16}$	$(C_8)^2$	$(C_4)^3$	$(C_2)^3 \times C_8$	$(C_2)^2 \times (C_4)^2$	$(C_2)^4 \times C_4$	$(C_2)^6$
$lb[G]$	9	9	9	10	10	11	12
$\Delta[G]$	10	10	11	11	12	12	14
$\partial[G]$	1,25	1,25	1,375	1,375	1,5	1,5	1,75
$G$	$(C_2)^2 \times C_{17}$	$C_2 \times C_4 \times C_9$	$(C_3)^2 \times C_8$	$C_2 \times (C_3)^2 \times C_4$	$(C_2)^3 \times (C_3)^2$	$(C_2)^3 \times C_9$	$C_3 \times (C_5)^2$
$lb[G]$	9	10	9	10	10	10	10
$\Delta[G]$	10	10	10	10	11	11	10
$\partial[G]$	1,2126...	1,1785...	1,1785...	1,1785...	1,2963...	1,2963...	1,1547...
$G$	$(C_2)^2 \times C_{19}$	$C_2 \times C_8 \times C_5$	$(C_4)^2 \times C_5$	$(C_2)^2 \times C_4 \times C_5$	$(C_2)^4 \times C_5$	$(C_9)^2$	$(C_3)^4$
$lb[G]$	10	10	10	10	11	10	10
$\Delta[G]$	11	11	11	12	12	11	12
$\partial[G]$	1,2617...	1,2298...	1,2298...	1,3416...	1,3416...	1,2222...	1,3333...
$G$	$(C_3)^2 \times C_9$	$C_3 \times C_{27}$	$(C_2)^2 \times C_3 \times C_7$	$(C_2)^3 \times C_{11}$	$C_2 \times C_4 \times C_{11}$	$C_2 \times (C_3)^2 \times C_5$	$(C_2)^2 \times C_{23}$
$lb[G]$	10	10	10	11	10	10	11
$\Delta[G]$	11	11	11	12	12	11	12
$\partial[G]$	1,2222...	1,2222...	1,2001...	1,2792...	1,2792...	1,1595...	1,2510...

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